October Math Gems

PROBLEM OF THE WEEK 23

§1 Problems

Problem 1.1. There are 45 rose and 18. These have to put into bouquets which contain both kinds. All bouquets should have the same number of flowers. Find the maximum number of bouquets and the number of flowers in them.

answer. Factorise 45 as 3*3*5 and 18 as 2*3*3
Take the highest common factor between the two numbers which is 9
Hence, the number of banquets that can formed is 9 and the total number of flowers are=2+5=7 □

Problem 1.2. Use Euclid's division lemma to show that the cube of any positive integer is one of these three forms $(9k, 9k+1, 9k+8)$ note that Euclid division algorithm $=$ $a = bq + r, 0 \leq r \leq b$
It indicate that for given positive integers a and b, there exist unique integers q and r satisfying the equation.

answer. Let a be any positive integer, and $b = 3$. Substitute $b = 3$ in equation (1)
$a = 3q + r$ where $0 \leq r \leq 3, r = 0, 1, 2$
If $r = 0, a = 3q$
Cube the value, we get
$a^3 = 27q^3$
$a^3 = 9(3q^3)$, where $k = 3q^3$ (2)
If $r = 1, a = 3q + 1$.Cube the value, we get $a^3 = (3q+1)^3 = (27q^3 + 27q^2 + 9q + 1) =$ $9(3q^3 + 3q^2 + q) + 1$ where $k = 3q^3 + 3q^2 + q(3)$

If $r = 2, a = 3q + 2$.Cube the value, we get $a^3 = (3q+2)^3 = (27q^3 + 53q^2 + 36q + 8) =$ $8(3q^3 + 6q^2 + 4q) + 8$ where $k = 3q^3 + 6q^2 + 4q(4)$
From equation 2,3 and 4,the cube of any positive integer is of the form $9m, 9m + 1or9m +$ 8. □

Problem 1.3. show that any positive even integer is one of these forms $(4q, 4q + 2)$ where q is the whole number.

answer. Let a be any positive integer and $b = 4$, then by Euclid's algorithm,

$a = 4q + r$, for some integer $q \geq 0$ and $r = 0, 1, 2, 3$ .So, $a = 4q$ or $4q + 1$ or $4q + 2$ or $4q + 3$ because $0 \leq r < 4$.

Now, $4q$ that is $(2 * 2q)$ is an even number.
Therefore, $4q + 1$ is an odd number.

Now, $4q + 2$ that is $2(2q + 1)$ which is also an even number.
Therefore, $(4q + 2) + 1 = 4q + 3$ is an odd number.

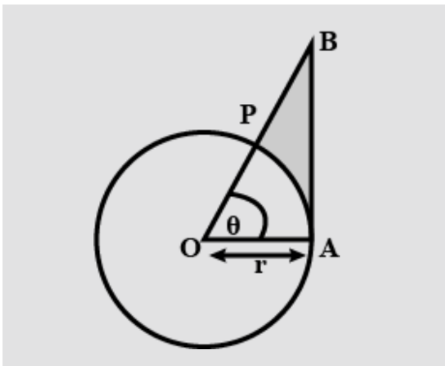
Hence, we can say that any even integer can be written in the form of 4q or 4q+2 where q is a whole number. □

Problem 1.4. If the polynomial $x^2 - 2x + k$ is a factor of$x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ find the value of a and k

answer. After dividing $x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ by $x^2 - 2x + k$
the remainder will be (2k-10)x+(10-a-8k+k^2)
As the second polynomial is a factor of the first polynomial.So, if we make the division the remainder will=0
$2k - 10 = 0, k = 5 ==>$ $(10 - a - 8k + k^2)$=0
$10 - a - 8 * 5 + 25 = 0 ==> -5 - a = 0$
$a = -5$ □

Problem 1.5. prove that the perimeter of shaded region is equal to

$$r[\tan\theta + \sec\theta + \frac{\pi\theta}{180} - 1]$$



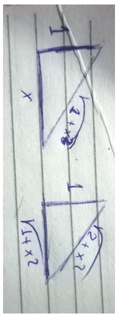
answer. IN triangle OAB, $\tan\theta = \frac{AB}{OA}$ where $OA = r$
$AB = r\ \tan\theta$
$\cos\theta = \frac{r}{OB}$, $OB = \frac{r}{\cos\theta} = r\sec\theta$
$OP + PB = OB$. Therefore, $PB = r\ \sec\theta - r$
arc $PA = \pi r * \frac{\theta}{180}$. You have $AB + PB + arc(PA)$= $r\tan\theta + *\frac{\theta}{180} + r\sec\theta - r =$ $r(\tan\theta + \sec\theta + \frac{\pi\theta}{180} - 1)$ □

Problem 1.6. Given that $3^x + 3^y = 10$,and $3^{xy} = 5$.Then, find

$$3^{x-y} + 3^{y-x}$$

answer. $\sin(cot^{-1}x) = \frac{1}{\sqrt{1+x^2}}$
$\cos(\tan^{-1}\frac{1}{\sqrt{1+x^2}}) = \sqrt{\frac{1+x^2}{2+x^2}}$ □

Problem 1.7. Evaluate the value of expression in terms of x, $\cos(\tan^{-1}(\sin(\cot^{-1}x)))$



answer. $n(n+1)\alpha = \frac{1}{\log_2 m} + \frac{1}{\log_4 m} + \frac{1}{\log_8 m} .... + \frac{1}{\log_{2^n} m}$
$\log_m 2 + 2\log_m 2 + 3\log_m 2 .... + n\log_m 2$
$(1+2+3....+n)\log_m 2 = \frac{n(n+1)}{2}\log_m 2 = n(n+1)\frac{1}{2}\log_m 2$
$n(n+1)\log_m \sqrt{2}. Then\ \alpha = \log_m \sqrt{2}$ □

Problem 1.8. $\frac{1}{\log_2 m} + \frac{1}{\log_4 m} + \frac{1}{\log_8 m} ...... + \frac{1}{\log_{2^n} m} = n(n+1)\alpha$
$\alpha =$

answer. $x^2 + ax + b = 0 ==> \alpha + \beta = -a$and $\alpha\beta = b$
$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}$
$= \frac{a^2-2b}{b^2}$ □

Problem 1.9. If α and β are the root of the equation $x^2 + ax + b = 0$, then $\frac{1}{\alpha^2} + \frac{1}{\beta^2} =$

answer. $x^2 + ax + b = 0 ==> \alpha + \beta = -a$and $\alpha\beta = b$
$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}$
$= \frac{a^2-2b}{b^2}$ □

Problem 1.10. Let P be a point inside triangle ABC such that the line AP is the bisector of the $\angle BAC$, BP is the bisector of $\angle ABC$ and CP is the bisector of $\angle ACB$. If $\angle APB = 110^o$, $\angle BPC = 120^o$ and $\angle CPA = 130^o$, find the angles of triangle ABC.

answer. 32) Let $\log(n)$ be the logarithm base 10. By change of base formula:

$$\frac{\log(22x)}{\log(20x)} = \frac{\log(202x)}{\log(2x)}$$

$$\Rightarrow \log 22x \cdot \log 2x = \log 20x \cdot \log 202x$$

$$\Rightarrow \log 22x \cdot \log 2x = (1 + \log 2x) \cdot \log 202x$$

$$\Rightarrow \log 22x \cdot \log 2x = \log 202x + \log 2x \cdot \log 202x$$

$$\Rightarrow \log 2x \cdot (\log 22x - \log 202x) = \log 202x$$

$$\Rightarrow \log 202x = \log 2x \cdot \log\left(\frac{11}{101}\right)$$

Thus, we have

$$\log_{20x}(22x) = \frac{\log(22x)}{\log(20x)} = \frac{\log(202x)}{\log(2x)} = \frac{\log 2x \cdot \log\left(\frac{11}{101}\right)}{\log 2x} = \log\left(\frac{11}{101}\right)$$

giving us an answer of $11 + 101 = \boxed{112}$. □

Problem 1.11. Let $ABCD$ be a quadrilateral inscribed in a circle such that $\angle ABC = 60°$. Prove that if $BC = CD$ then $AB = CD + DA$.

Problem 1.12. There is a positive real number x not equal to either $\frac{1}{20}$ or $\frac{1}{2}$ such that

$$\log_{20x}(22x) = \log_{2x}(202x).$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}(\frac{m}{n})$, where m and n are relatively prime positive integers. Find $m + n$.

answer. 32) Let $\log(n)$ be the logarithm base 10. By change of base formula:

$$\frac{\log(22x)}{\log(20x)} = \frac{\log(202x)}{\log(2x)}$$

$$\Rightarrow \log 22x \cdot \log 2x = \log 20x \cdot \log 202x$$

$$\Rightarrow \log 22x \cdot \log 2x = (1 + \log 2x) \cdot \log 202x$$

$$\Rightarrow \log 22x \cdot \log 2x = \log 202x + \log 2x \cdot \log 202x$$

$$\Rightarrow \log 2x \cdot (\log 22x - \log 202x) = \log 202x$$

$$\Rightarrow \log 202x = \log 2x \cdot \log\left(\frac{11}{101}\right)$$

Thus, we have

$$\log_{20x}(22x) = \frac{\log(22x)}{\log(20x)} = \frac{\log(202x)}{\log(2x)} = \frac{\log 2x \cdot \log\left(\frac{11}{101}\right)}{\log 2x} = \log\left(\frac{11}{101}\right)$$

giving us an answer of $11 + 101 = \boxed{112}$. □

Problem 1.13. Find the number of real roots of the equation

$$e^{4x} - e^{3x} - 4e^{2x} - e^x + 1$$

Problem 1.14. Find the sum of the following infinite series

$$1 + \frac{3}{\pi} + \frac{3}{\pi^2} + \frac{3}{\pi^3} + \dots$$

Problem 1.15. There are real numbers a, b, c, and d such that -20 is a root of $x^3 + ax + b$ and -21 is a root of $x^3 + cx^2 + d$. These two polynomials share a complex root $m + \sqrt{n} \cdot i$, where m and n are positive integers and $i = \sqrt{-1}$. Find $m + n$.

answer. 35)Since we know each polynomial has a real root and share the complex root $m + \sqrt{n}i$, the other root must be the complex conjugate which is $m - \sqrt{n}i$.

Applying Vieta's on the equation $x^3 + ax + b$, we find that the sum of the roots is 0. Therefore,

$$-20 + (m + \sqrt{n}i) + (m - \sqrt{n}i) = 0$$

$$2m = 20$$

$$m = 10.$$

Applying Vieta's on the equation $x^3 + cx^2 + d$, we find that the sum of the product of the roots taken in pairs of 2 is 0. Therefore,

$$(-21)(m + \sqrt{n}i) + (-21)(m - \sqrt{n}i) + (m + \sqrt{n}i)(m - \sqrt{n}i) = 0$$
$$-21m - 21\sqrt{n}i - 21m + 21\sqrt{n}i + m^2 + n = 0.$$

We know that m is 10, so

$$-42(10) + 100 + n = 0$$
$$n = 320.$$

Therefore, $m + \sqrt{n}i = 10 + \sqrt{320}$, so $m + n = 330$ □

Problem 1.16. Let a, b, c, and d be real numbers that satisfy the system of equations

$$\begin{aligned} a + b &= -3 \\ ab + bc + ca &= -4 \\ abc + bcd + cda + dab &= 14 \\ abcd &= 30. \end{aligned}$$

There exist relatively prime positive integers m and n such that

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}.$$

Find $m + n$.

answer. 36) Let $d = \frac{30}{abc}$. Equation (3) becomes :

$$abc + \frac{30(ab + bc + ca)}{abc} = abc - \frac{120}{abc} = 14$$

Hence $(abc)^2 - 14(abc) - 120 = 0$. Solving the above we get $abc = -6$ or $abc = 20$. $ab + bc + ca = ab - 3c = -4 \Longrightarrow ab = 3c - 4$. If $abc = -6$, then $c(3c - 4) = -6$. This gives does not give real solutions for c. Hence $abc = 20$. $c(3c - 4) = 20 \implies c = -2$ or $c = \frac{10}{3}$. If $c = 10/3$, then the system $a + b = -3$ and $ab = 6$. Does not have real solutions for both a and b. Hence $c = -2$.

$$a^2 + b^2 + c^2 + d^2 = 9 + \frac{9}{4} - 2ab + 4 = 9 + 20 + 4 + \frac{9}{4} = \frac{141}{4}$$

The answer is $\boxed{145}$. □

Problem 1.17. Two infinite geometric series have the same sum. The first term of the first series is 1, and the first term of the second series is 4. The fifth terms of the two series are equal. The sum of each series can be written as $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$

Problem 1.18. By using heron's formula solve the following question
The side lengths of a scalene triangle are the roots of the polynomial

$$x^3 - 20x^2 + 131x - 281.3$$

Find the square of the area of the triangle.

Problem 1.19. Known that

$$\frac{\sin^4(x)}{2} + \frac{\cos^4(x)}{3} = \frac{1}{5}$$

Then, $\tan^2(x) =$

Problem 1.20. If $a\cos(A) + b\sin(A) = a\cos(B) + b\sin(B)$, show that

$$\cos^2(\frac{A+B}{2}) - \sin^2(\frac{A+B}{2}) = \frac{a^2 - b^2}{a^2 + b^2}$$

answer. 40)$cos^2\frac{A+B}{2} - sin^2\frac{A+B}{2} = cos(A+B)$ $acosA + bsinA = acosB + bsinB$ Squaring both sides we get, $a^2cos^2A + b^2sin^2A + absin2A = a^2cos^2B + b^2sin^2B + absin2B$
$a^2(cos^2A - cos^2B) + b^2(sin^2A - sin^2B) + ab(sin2A - sin2B) = 0$
$(b^2 - a^2)(sin^2A - sin^2B) + 2abcos(A+B)sin(A-B) = 0$
$(b^2 - a^2)sin(A-B)sin(A+B) + 2abcos(A+B)sin(A-B) = 0$
$2abcos(A+B) = (a^2 - b^2)sin(A+B)$
$tan(A+B) = \frac{2ab}{a^2-b^2}$
Hence, $cos(A+B) = \frac{a^2-b^2}{a^2+b^2}$ □